# 4. Fourier Series

## 4.1 Periodic Functions and Trignometric Series

### 4.1.1 Periodic Functions

A function $f(x)$ is called periodic if it is defined for all (except $\tan x$ at $\pm\pi/2$, $\pm 3\pi/2$……, whose period is $\pi$) real $x$ and if there is some positive number $p$ such that

$$f(x+p)=f(x) \quad \text{for all } x.$$

The number $p$ is called **period** of $f(x)$. The graph of such function is obtained by periodic repetition of its graph in any interval of length $p$.

**NOTE:**

(i) Familiar periodic functions are sine and cosine functions.

(ii) The function $f=\text{constant}$ is also a periodic function.

(iii) The functions that are not periodic are $x$, $x^2$, $x^3$, $e^x$, $\cosh x$, $\ln x$ etc.

(iv) $\because f(x+2p)=f[(x+p)+p]=f(x+p)=f(x)$.

Thus for any integer $n$, $f(x+np)=f(x)$. Hence $2p, 3p,\ldots\ldots$ are also period of $f(x)$.

(v) If $f(x)$ and $g(x)$ have period $p$, then the function

$$h(x)=af(x)+bg(x) \qquad (a, b \text{ constants})$$

has also period $p$.

**Fundamental Period**

If a periodic function $f(x)$ has a smallest period $p\,(>0)$, this is often called the fundamental period of $f(x)$.

**Example:**

(i) For $\sin x$ and $\cos x$ the fundamental period is $2\pi$.

(ii) For $\sin 2x$ and $\cos 2x$ the fundamental period is $\pi$.

(iii) For $\tan x$ and $\cot x$ the fundamental period is $\pi$.

(iv) A functions without fundamental period is $f=\text{constant}$.

### 4.1.2 Trignometric Series

Let's represent various functions of period $p = 2\pi$ in terms of simple functions

$1, \quad \cos x, \sin x, \quad \cos 2x, \sin 2x, \ldots\ldots \cos nx, \sin nx, \ldots\ldots$

These functions have period $2\pi$. Figure below shows the first few of them.

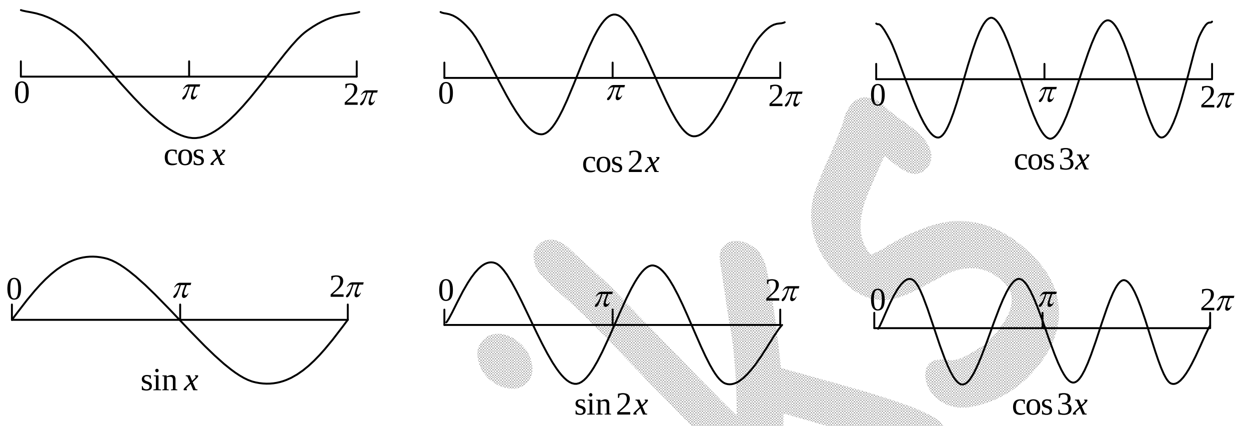



Figure: Cosine and sine functions having the period $2\pi$

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots\ldots\ldots$$

Where $a_0, a_1, a_2 \ldots b_1, b_2 \ldots$ are real constants. Such a series is called trigonometric seris and the $a_n$ and $b_n$ are called the coefficient of the series. Thus we may write series

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

We see that each term of the series has the period $2\pi$. Hence if the series converges, its sum will be a function of period $2\pi$.

**NOTE:**

The trignometric series can be used for representing any practically important periodic function $f$, simple or complicated, of any period $p$. This series will then be called the Fourier series of $f$.

## 4.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series whose coefficients are determined from $f(x)$ by the "Euler Formulas".

### 4.2.1 Euler Formulas for the Fourier Coefficients

Let us assume that $f(x)$ is periodic function of period $2\pi$ and is integrable over a period. Let us further assume that $f(x)$ can be represented by a trigonometric series, (assume that this series converges and has $f(x)$ as its sum)

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \text{ ............(1)}$$

**Determination of constant $a_0$ :**

From equation (1), we have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nxdx + b_n\int_{-\pi}^{\pi}\sin nxdx\right)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 \qquad \because \int_{-\pi}^{\pi}\cos nxdx = 0, \quad \int_{-\pi}^{\pi}\sin nxdx = 0$$

Thus $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \text{ ................(2)}$$

**Determination of constant $a_n$ :**

Multiply equation (1) by $\cos mx$, where $m$ is any fixed positive integer, then

$$\int_{-\pi}^{\pi} f(x)\cos mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mxdx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mxdx = a_0\int_{-\pi}^{\pi}\cos mxdx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mxdx + b_n\int_{-\pi}^{\pi}\sin nx\cos mxdx\right)$$

$\because \int_{-\pi}^{\pi} \cos mx dx = 0$,

$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx = 0$, alwyas

$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$.

$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx dx = a_n \pi$ when $n = m$

Thus $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx}$ ................(3)

**Determination of constant $b_n$ :**

Multiply equation (1) by $\sin mx$, where $m$ is any fixed positive integer, then

$\int_{-\pi}^{\pi} f(x)\sin mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mx dx$

$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mx dx = a_0\int_{-\pi}^{\pi} \sin mx dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi} \cos nx \sin mx dx + b_n\int_{-\pi}^{\pi} \sin nx \sin mx dx\right)$

$\because \int_{-\pi}^{\pi} \sin mx dx = 0$,

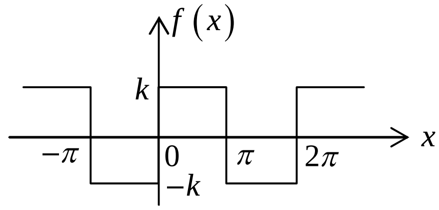
$\int_{-\pi}^{\pi} \cos nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(m-n)x dx = 0$, always.

$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$.

$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin nx dx = b_n \pi$ when $n = m$

Thus $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx}$ ................(4)

**Example:** Find the Fourier coefficient of the periodic function $f(x)$ as shown in figure:




$$f(x)=\begin{cases}-k & \text{if} \quad -\pi<x<0 \\ k & \text{if} \quad 0<x<\pi\end{cases} \quad \text{and} \quad f(x+2\pi)=f(x)$$

Hence show that: $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$

**Solution:**

Let $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,dx+\int_{0}^{\pi}(k)\,dx\right]=\frac{1}{2\pi}\left[[-kx]_{-\pi}^{0}+[kx]_{0}^{\pi}\right]=0$$

This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ to $\pi$ is zero.

$$\because a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$\Rightarrow a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nxdx+\int_{0}^{\pi}(k)\cos nxdx\right]=\frac{1}{\pi}k\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0}+\left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right]=0$$

$$\because b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

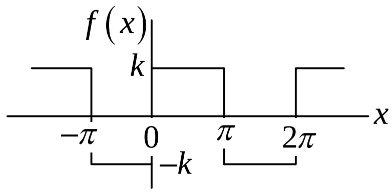
$$\Rightarrow b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nxdx+\int_{0}^{\pi}(k)\sin nxdx\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0}-\left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right]=\frac{1}{\pi}k\left[\frac{1}{n}-\frac{(-1)^n}{n}-\frac{(-1)^n}{n}+\frac{1}{n}\right]=\frac{1}{\pi}k\left[\frac{2}{n}-\frac{2(-1)^n}{n}\right]$$

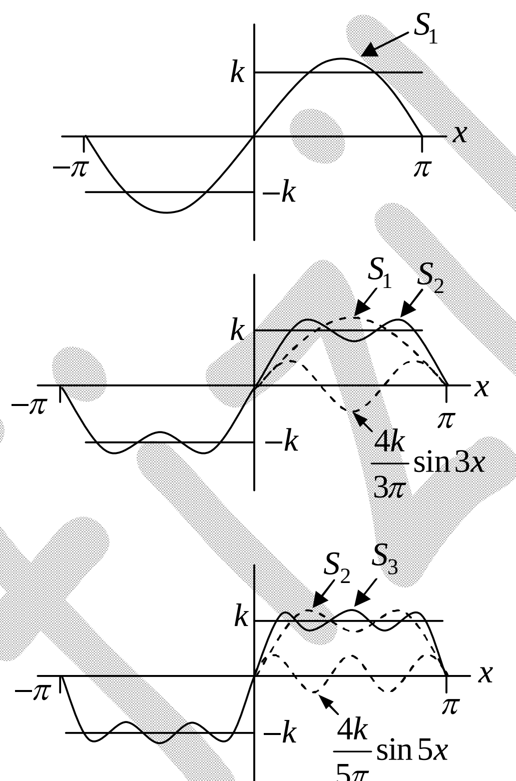
If $n$ is even $b_n=0$ and If $n$ is odd $b_n=\frac{4k}{n\pi}$.

Thus Fourier series is $f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$

The partial sums are $S_1=\frac{4k}{\pi}\sin x$, $S_2=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x\right)$, etc




Figure(a): The given function $f(x)$ (Period square wave)




Figure(b): The first three partial sums of the corresponding Fourier series

**NOTE:**

The above graph seems to indicate that the series is convergent and has the sum $f(x)$, the given function. Notice that at $x=0$ and $x=\pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and $+k$ of our function.

Assuming that $f(x)$ is the sum of the series and setting $x=\frac{\pi}{2}$, we have

$$f\left(\frac{\pi}{2}\right)=k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\ldots\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+\ldots\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

$$\Rightarrow \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

### 4.2.2 Orthogonality of the Trignometric System

The trigonometric system

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ldots\ldots\ldots \cos nx,\ \sin nx,\ldots\ldots\ldots$$

is orthogonal on the interval $-\pi \le x \le \pi$ (hence on any interval of length $2\pi$, because of periodicity). Thus for any integer $m$ and $n$ we have

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=0 \qquad (m\neq n)$$

and

$$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=0 \qquad (m\neq n)$$

and for any integer $m$ and $n$ (including $m=n$) we have

$$\int_{-\pi}^{\pi}\cos mx\sin nx\,dx=0$$

### 4.2.3 Convergence and Sum of Fourier Series

If a periodic function $f(x)$ with period $2\pi$ is piecewise continuous in the interval $-\pi \le x \le \pi$ and has left hand and right hand derivative at each point of that interval, then the Fourier series of $f(x)$ with coefficient $a_0, a_n, b_n$ is convergent.

Its sum is $f(x)$, except at a point $x_0$ at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limit of $f(x)$ at $x_0$.

**NOTE:**

(i) The left-hand limit of $f(x)$ at $x_0$ is $f(x_0-0)=\lim\limits_{h\to 0} f(x_0-h)$.

The right-hand limit of $f(x)$ at $x_0$ is $f(x_0+0)=\lim\limits_{h\to 0} f(x_0+h)$.

(ii) The left-hand derivative of $f(x)$ at $x_0$ is

$$\lim_{h\to 0}\frac{f(x_0-h)-f(x_0-0)}{-h}.$$

The right-hand derivative of $f(x)$ at $x_0$ is

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0+0)}{h}.$$

(iii) If $f(x)$ is continuous at $x_0$, then

$$\lim_{h\to 0}\frac{f(x_0-h)-f(x_0-0)}{-h}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0+0)}{h}=f(x_0)$$

**Example:** The square wave in previous example has a jump at $x=0$. Its left-hand limit there is $-k$ and its left-hand limit there is $+k$. Hence average of these limits is $0$. Thus Fourier series converge to this value at $x=0$ because then all its terms are $0$. Similarly for the other jump we can verify this.

**4.3 Function of Any Period $p = 2L$**

The functions considered so far had period $2\pi$, for simplicity. The transition from $p = 2\pi$ to $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis.

Fourier series of a function $f(x)$ of period $p = 2L$, is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where Fourier coefficients of $f(x)$ are

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Let $v = \frac{\pi x}{L} \Rightarrow x = \frac{Lv}{\pi}$ and $dv = \frac{\pi dx}{L}$. Also $x = \pm L$ corresponds to $v = \pm\pi$.

Thus $f(x) = g(v)$ has period $2\pi$.

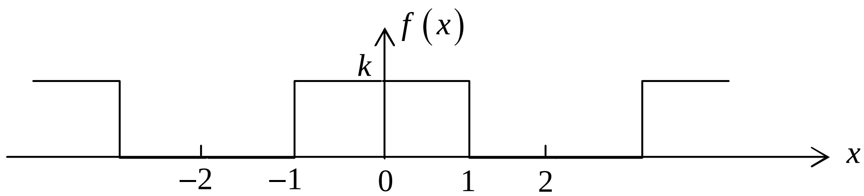
Hence we can verify that

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv$$

**Example:** Find the Fourier series of the function $f(x)$ as shown in figure:




$$f(x)=\begin{cases}0 & \text{if} \quad -2<x<-1 \\ k & \text{if} \quad -1<x<1 \\ 0 & \text{if} \quad 1<x<2\end{cases} \qquad p=2L=4,\ L=2$$

**Solution:** Let $f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$

$$\because a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$$

$$\Rightarrow a_0=\frac{1}{4}\int_{-2}^{2}f(x)\,dx=\frac{1}{4}\int_{-1}^{1}k\,dx=\frac{k}{2}$$

$$\because a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi x}{2}dx=\frac{1}{2}\int_{-1}^{1}k\cos\frac{n\pi x}{2}dx$$

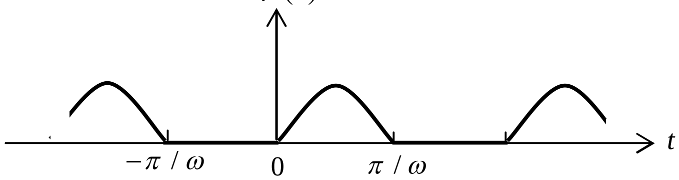
$$\Rightarrow a_n=\frac{k}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-1}^{1}=\frac{2k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases}0, & n=2,4,6.....\\ \frac{2k}{n\pi}, & n=1,5,9.....\\ -\frac{2k}{n\pi}, & n=3,7,11.....\end{cases}$$

$$\because b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin\frac{n\pi x}{2}dx=\frac{1}{2}\int_{-1}^{1}k\sin\frac{n\pi x}{2}dx=\frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_{-1}^{1}=0$$

Thus Fourier series is $f(x)=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi}{2}x-\frac{1}{3}\cos\frac{3\pi}{2}x+\frac{1}{5}\cos\frac{5\pi}{2}x-+....\right)$

**Example:** A sinusoidal voltage $E_0 \sin \omega t$ where $t$ is time is passed through half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function




$$f(t) = \begin{cases} 0 & \text{if} \quad -\frac{\pi}{\omega} < t < 0 \\ E_0 \sin \omega t & \text{if} \quad 0 < t < \frac{\pi}{\omega} \end{cases} \qquad p = 2L = 2\frac{\pi}{\omega}, \; L = \frac{\pi}{\omega}$$

**Solution:** Let $f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$

$$\because a_0 = \frac{1}{2L} \int_{-L}^{L} f(t)\, dt$$

$$\Rightarrow a_0 = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t)\, dt = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E_0 \sin \omega t dt$$

$$\Rightarrow a_0 = \frac{\omega}{2\pi} \left[ -\frac{E_0}{\omega} \cos \omega t \right]_0^{\pi/\omega} = \frac{E_0}{\pi}$$

$$\because a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt$$

$$\Rightarrow a_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \cos n\omega t dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} E_0 \sin \omega t \cos n\omega t dt = \frac{E_0 \omega}{2\pi} \int_0^{\pi/\omega} 2 \sin \omega t \cos n\omega t dt$$

$$\Rightarrow a_n = \frac{E_0 \omega}{2\pi} \int_0^{\pi/\omega} \left[ \sin(1+n)\omega t + \sin(1-n)\omega t \right] dt$$

For $n = 1$

$$a_1 = \frac{E_0 \omega}{2\pi} \int_0^{\pi/\omega} \sin 2\omega t dt = \frac{E_0 \omega}{2\pi} \left[ \frac{-\cos 2\omega t}{2\omega} \right]_0^{\pi/\omega} = 0$$

For $n = 2, 3, 4....$

$$\Rightarrow a_n = \frac{E_0\omega}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}\right] = \begin{cases} 0 & n = 3,5,7... \\ \dfrac{2E_0}{\pi(1-n^2)} & n = 2,4,6.... \end{cases}$$

$$\because b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E_0 \sin\omega t \sin n\omega t dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega} 2\sin\omega t\sin n\omega t dt$$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[-\cos(1+n)\omega t + \cos(1-n)\omega t\right]dt$$

For $n = 1$

$$b_1 = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[1 - \cos 2\omega t\right]dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{\pi/\omega} = \frac{E_0\omega}{2\pi}\frac{\pi}{\omega} = \frac{E_0}{2}$$

For $n = 2, 3, 4....$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\left[-\frac{\sin(1+n)\omega t}{(1+n)\omega} + \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E_0}{2\pi}\left[\frac{-\sin(1+n)\pi}{(1+n)} + \frac{\sin(1-n)\pi}{(1-n)}\right] = 0$$

Thus Fourier series

$$f(t) = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty} a_n\cos n\omega t = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty}\frac{2E_0}{\pi(1-n^2)}\cos n\omega t$$

$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{\cos 2\omega t}{3} + \frac{\cos 4\omega t}{15} + ...\right]$$

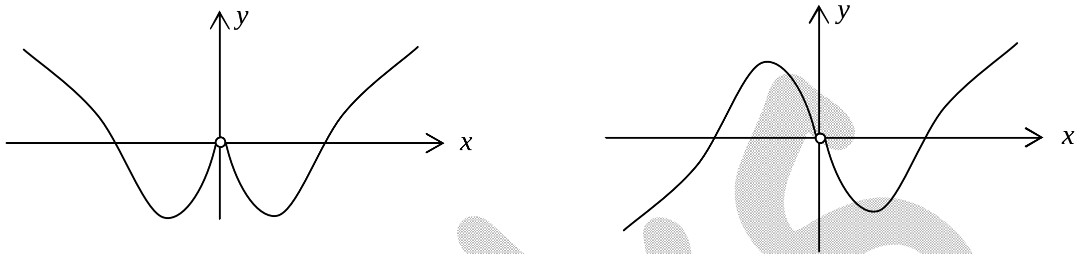
$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right]$$

## 4.4 Even and Odd functions and Half-Range Expansion

### 4.4.1 Even and Odd function

A function $g(x)$ is said to be even if $g(-x) = g(x)$, so that its graph is symmetrical with respect to vertical axis.

A function $h(x)$ is said to be even if $h(-x) = -h(x)$.




(a) Even function (b) Odd function

Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx \qquad \text{for even } g(x)$$

$$\int_{-L}^{L} h(x)\,dx = 0 \qquad \text{for odd } h(x)$$

**Fourier Cosine Series and Fourier Sine Series**

Fourier series of an **even** function of period $2L$, is a **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad \text{.....................(1)}$$

with coefficients( note integration from 0 to $L$ )

$$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)\,dx}\;, \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx} \qquad \text{...............(2)}$$

Fourier series of an odd function of period $2L$, is a **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad \text{.....................(3)}$$

with coefficients

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx} \qquad \text{...............(4)}$$

**NOTE:**

(i) For even function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$.

(ii) For odd function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = 0, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 0$$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$.

**The Case of Period $2\pi$**

If $L = \pi$, and $f(x)$ is even function then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{....................}(1')$$

with coefficients

$$\boxed{a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx}, \quad \boxed{a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx} \qquad \text{...............}(2')$$

If $f(x)$ is odd function then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{...................}(3')$$

with coefficients

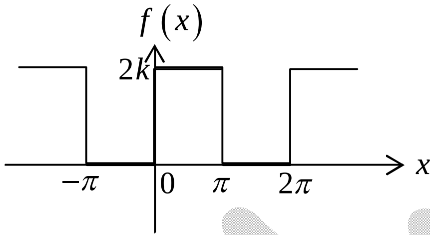
$$\boxed{b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx} \qquad \text{...............}(4')$$

**Sum and Scalar Multiple**

(a) The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of $f_1$ and $f_2$.

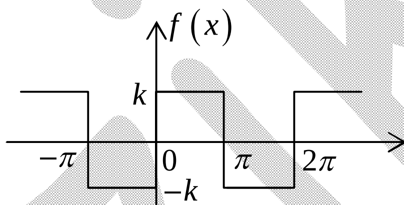
(b) The Fourier coefficients of $cf$ are $c$ times the corresponding Fourier coefficients of $f$.

**Example:** Find the Fourier series of the periodic function $f(x)$ as shown in figure:




**Solution:**

We have already calculated the Fourier series of the periodic function $f(x)$ as shown in figure:




The Fourier series is $f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$.

The function given in the problem can be obtained by adding $k$ to the above function. Thus the Fourier series of a sum $k + f(x)$ are the sums of the corresponding Fourier series of $k$ and $f(x)$.

The Fourier series is $f(x) = k + \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$.

**Example:** Find the Fourier series of the periodic function

$$f(x) = x + \pi \ ; \quad (-\pi < x < \pi) \text{ having period } 2\pi$$

**Solution:**

Let $f(x) = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$.

The Fourier coefficient of $f_2 = \pi$ is $a_0 = \pi$, $a_n = 0$ and $b_n = 0$.

The Fourier coefficient of $f_1 = x$ is

$$a_0 = 0,\ a_n = 0 \text{ and } b_n = -2\frac{(-1)^n}{n}; \quad n = 1, 2, 3.....$$

Thus the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x...\right].$$

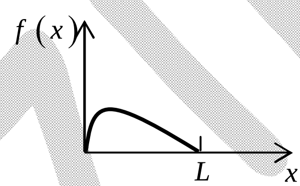
### 4.4.2 Half-Range Expansion

Half-range expansions are Fourier series. The idea is simple and useful. We could extend $f(x)$ as a function of period $L$ and develop the extended function into a Fourier series. But this series would in general contain both cosine and sine terms.

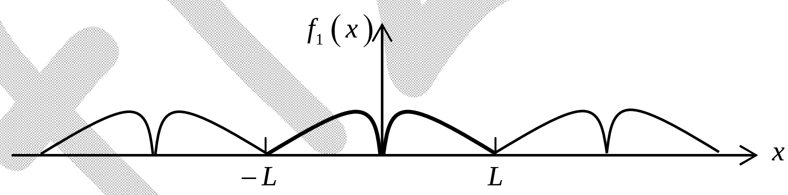
We can do better and get simpler series. For our given function $f(x)$ we can calculate **Fourier cosine series** coefficient $(a_0$ and $a_n)$. This is the even periodic extension $f_1(x)$ of $f(x)$ in figure $(b)$.

For our given function $f(x)$ we can calculate **Fourier sine series** coefficient $(b_n)$. This is the odd periodic extension $f_2(x)$ of $f(x)$ in figure $(c)$.

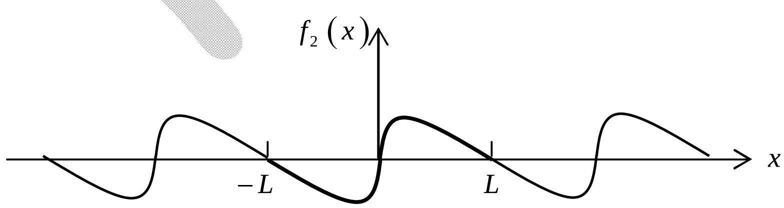
Both extensions have period $2L$. Note that $f(x)$ is given only on half the range, half the interval of periodicity of length $2L$.




(a) The given function



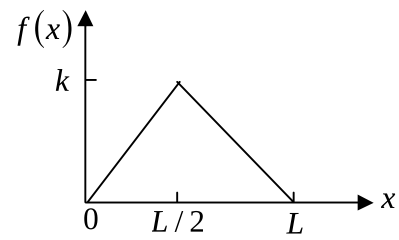

(b) $f(x)$ extended as an even periodic function of period $2L$




(c) $f(x)$ extended as an odd periodic function of period $2L$

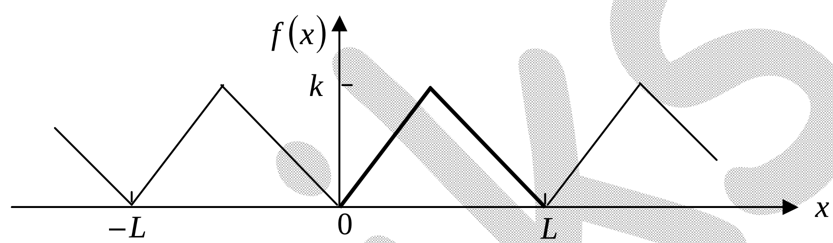
**Example:** Find the two half-range expansion of the function $f(x)$ as shown in figure below.

$$f(x)=\begin{cases}\frac{2k}{L}x & \text{if} \quad 0<x<\frac{L}{2}\\ \frac{2k}{L}(L-x) & \text{if} \quad \frac{L}{2}<x<L\end{cases}$$




**Solution:**

**Even periodic extension**




$$\because a_0=\frac{1}{L}\int_0^L f(x)\,dx$$

$$\Rightarrow a_0=\frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}}x\,dx+\frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\,dx\right]=\frac{k}{2}$$

$$\because a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$\Rightarrow a_n=\frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}}x\cos\frac{n\pi x}{L}dx+\frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right]$$

Let us calculate the integral

$$\int_0^{\frac{L}{2}}x\cos\frac{n\pi x}{L}dx=\frac{L}{n\pi}\left[x\sin\frac{n\pi x}{L}\right]_0^{L/2}+\left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_0^{L/2}$$

$$\Rightarrow\int_0^{\frac{L}{2}}x\cos\frac{n\pi x}{L}dx=\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)$$

and $\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx=\frac{L}{n\pi}\left[(L-x)\sin\frac{n\pi x}{L}\right]_{L/2}^{L}-\left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_{L/2}^{L}$

$$\Rightarrow\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx=-\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}-\frac{L^2}{n^2\pi^2}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)$$

$$\Rightarrow a_n = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right] = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

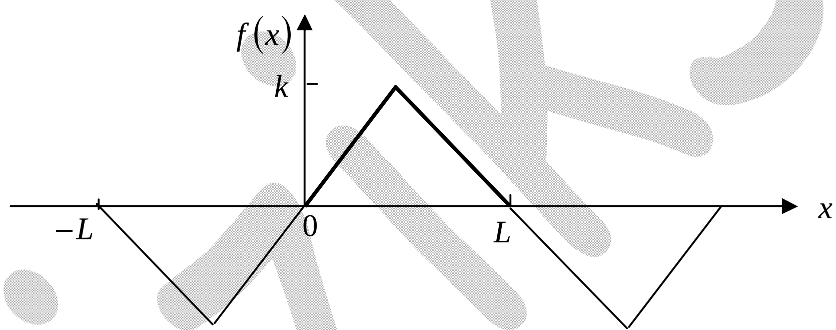
$$\Rightarrow a_2 = -\frac{16k}{2^2\pi^2}, a_6 = -\frac{16k}{6^2\pi^2}, a_{10} = -\frac{16k}{10^2\pi^2}, \ldots\ldots\ldots\ldots \text{ and } a_n = 0 \text{ if } n \neq 2,6,10.....$$

Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \frac{1}{10^2}\cos\frac{10\pi x}{L} + .....\right)$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

**Odd periodic extension**




$$\because b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^{\frac{L}{2}} f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right]$$

Let us calculate the integral

$$\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-x\cos\frac{n\pi x}{L}\right]_0^{L/2} + \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_0^{L/2}$$

$$\Rightarrow \int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and $\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-(L-x)\cos\frac{n\pi x}{L}\right]_{L/2}^{L} - \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L}$

$$\Rightarrow \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L} - + .....\right)$$

This Fourier sine series represents the odd periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

**4.5 Complex Fourier series**

The Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \text{............(1)}$$

can be written in complex form, which sometimes simplifies calculations.

$\because e^{inx} = \cos nx + i\sin nx$ and $e^{-inx} = \cos nx - i\sin nx$

$\Rightarrow \cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$ and $\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$

Thus $a_n\cos nx + b_n\sin nx = \frac{1}{2}a_n\left(e^{inx} + e^{-inx}\right) + \frac{1}{2i}b_n\left(e^{inx} - e^{-inx}\right)$

$$\Rightarrow a_n\cos nx + b_n\sin nx = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$$

Lets take $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$ and $\frac{1}{2}(a_n + ib_n) = k_n$, then (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + k_n e^{-inx}\right) \text{............(2)}$$

where coefficients $c_0$, $c_n$ and $k_n$ are given by

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

We can also write (2) as (take $k_n = c_{-n}$)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx; \quad n = 0, \pm1, \pm2.......$$

This is so called complex form of the Fourier series or, complex Fourier series of $f(x)$.

The $c_n$ are called complex Fourier series coefficients of $f(x)$.

**For a function of period 2*L***

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \quad \text{where } c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx; \quad n = 0, \pm1, \pm2.......$$

**Example:** Find the complex Fourier series of the periodic function

$$f(x) = e^x; \quad (-\pi < x < \pi), \qquad \text{having period } 2\pi .$$

**Solution:** Let Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\because c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$\Rightarrow c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx = \frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)x}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right]$$

$$\Rightarrow c_n = \frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right] = \frac{1}{2\pi}\left(\frac{1+in}{1+n^2}\right)\left[e^{\pi} - e^{-\pi}\right](-1)^n \qquad \because e^{\pm in\pi} = (-1)^n$$

$$\Rightarrow c_n = \left(\frac{1+in}{1+n^2}\right)\frac{\sinh\pi}{\pi}(-1)^n \qquad \because \sinh\pi = \frac{e^{\pi} - e^{-\pi}}{2}$$

Thus Fourier series is $$f(x) = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{1+in}{1+n^2}\right)e^{inx} \;............(1)$$

Let us derive the real Fourier series

$\because (1+in)e^{inx} = (1+in)(\cos nx + i\sin nx) = (\cos nx - n\sin nx) + i(\cos nx + \sin nx)$

$\because n$ varies from $-\infty$ to $+\infty$, equation (1) has corresponding term with $-n$ instead of $n$.

Thus

$\because (1-in)e^{-inx} = (1-in)(\cos nx - i\sin nx) = (\cos nx - n\sin nx) - i(\cos nx + \sin nx)$

Let's add these two expressions;

$$(1+in)e^{inx} + (1-in)e^{-inx} = 2(\cos nx - n\sin nx), \qquad n = 1,2,3......$$

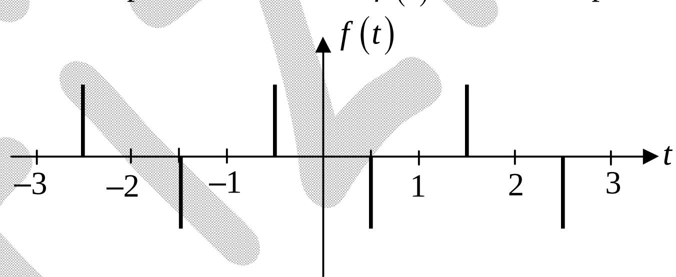
For $n = 0$, $\quad (-1)^n \left(\frac{1+in}{1+n^2}\right)e^{inx} = 1$

Thus

$$f(x) = \frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{(\cos nx - n\sin nx)}{1+n^2}$$

$$f(x) = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - +...\right]$$

**Example:** Consider the periodic function $f(t)$ with time period $T$ as shown in the figure below.




The spikes, located at $t = \frac{1}{2}(2n-1)$, where $n = 0, \pm 1, \pm 2, ...$, are Dirac-delta function of strength $\pm 1$. Find the amplitudes $a_n$ in the Fourier expansion of

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \text{int}/T} \quad .$$

**Solution:**

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$ where $c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx; \quad n = 0, \pm 1, \pm 2.......$

and Range: $\left[-1,1\right]$ hence $2L=2$.

Comparing with $f\left(t\right)=\sum_{n=-\infty}^{\infty}a_n e^{2\pi \mathrm{int}/T}$ , $L=\frac{T}{2}=1\Rightarrow T=2$.

$$\Rightarrow f\left(t\right)=\sum_{n=-\infty}^{\infty}a_n e^{i\pi nt}\Rightarrow a_n=\frac{1}{2L}\int_{-L}^{L}f\left(t\right)e^{-i\pi nt}dt$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-1}^{1}f\left(t\right)e^{-i\pi nt}dt=\frac{1}{2}\int_{-1}^{1}\left[\delta\left(t+\frac{1}{2}\right)-\delta\left(t-\frac{1}{2}\right)\right]e^{-i\pi nt}dt$$

$$\Rightarrow a_n=\frac{1}{2}\left[e^{-i\pi n(-1/2)}-e^{-i\pi n(1/2)}\right]=\frac{1}{2}\left[e^{i\pi n/2}-e^{-i\pi n/2}\right]=i\sin\frac{n\pi}{2}$$

**4.6 Approximation by Trignometric Polynomials**

Fourier series have major applications in approximation theory, that is, the approximation of functions by simpler functions.

Let $f\left(x\right)$ be a periodic function, of period $2\pi$ for simplicity that can be represented by a Fourier series. Then the $N^{th}$ partial sum of the series is an approximation to $f\left(x\right)$:

$$f\left(x\right)\approx a_0+\sum_{n=1}^{N}\left(a_n\cos nx+b_n\sin nx\right)\ldots\ldots\ldots\ldots(1)$$

We have to see whether (1) is the "best" approximation to $f$ by a trignometric polynomial of degree $N$ , that is , by a function of the form

$$F\left(x\right)=A_0+\sum_{n=1}^{N}\left(A_n\cos nx+B_n\sin nx\right)\ldots\ldots\ldots\ldots(2),$$

where "best" means that the "error" of approximation is minimum.

The **total square error** of $F$ relative to $f$ on the interval $-\pi\le x\le\pi$ is given by

$$E=\int_{-\pi}^{\pi}\left(f-F\right)^2 dx \qquad \text{clearly } E\ge 0.$$

The function $F$ is a good approximation to $f$ but $\left|f-F\right|$ is large at a point of discontinuity $x_0$.

### 4.6.1 Minimum square error

The total square error of $F$ relative to $f$ on the interval $-\pi \le x \le \pi$ is minimum if and only if the coefficients of $F(x)$ are the Fourier coefficients of $f(x)$. This minimum value $E^*$ is given by

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right] \quad \ldots\ldots\ldots\ldots(3),$$

From (3) we can see that $E^*$ cannot increase as $N$ increases, but may decrease. Hence with increasing $N$ the partial sums of the Fourier series of $f$ yields better and better approximations to $f$.

### 4.6.2 Parseval's Identity

Since $E^* \ge 0$ and equation (3) holds for every $N$, we obtain

$$2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx .$$

Now Parseval's Identity is

$$2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$$

**Example:** Compute the total square error of $F$ with $N = 3$ relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

on the interval $-\pi \le x \le \pi$.

**Solution:**

Fourier coefficients are $a_0 = \pi$, $a_n = 0$ and $b_n = -\frac{2}{n}\cos n\pi$.

Its Fourier series is given by

$$F(x) = \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x .$$

Hence

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right] = \int_{-\pi}^{\pi}\left(x+\pi\right)^2 dx - \pi\left[2\pi^2 + 2^2 + 1^2 + \left(\frac{2}{3}\right)^2\right]$$

$$E^* = \frac{8}{3}\pi^3 - \pi\left[2\pi^2 + \frac{49}{9}\right] \approx 3.567$$

Although $\left|f\left(x\right) - F\left(x\right)\right|$ is large at $x = \pm\pi$, where $f$ is discontinuous, $F$ approximates $f$ quite well on the whole interval.